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# *LAW OF ERROR IN THE POSITION OF A POINT IN SPACE.*

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THE law of facility of error in the observed position of a point varies according as the point is supposed to lie on a given straight line, or in a given plane, or in unrestricted space of three dimensions. Most quantities which are the subjects of observation are capable of having their magnitude represented by the length of a straight line, and their errors are represented by the errors in the position of the point which forms one extremity of the line, the other extremity being fixed. Taking this line as an axis of $X$, the probability $y$ that the point of error will fall within any arbitrary but very small interval $dx$ on this line, is a function of the form

$$y = \frac{hdx}{\sqrt{\pi}}e^{-h^2x^2}, \tag{1}$$

where $x$ is the distance from the most probable position of the point to the middle of the interval $dx$, and $h$ is a constant. The most probable position of the point, and the value of $h$, are to be determined by known methods. If an observed point may be anywhere on a given plane, as in the case of shot-marks on a target, the probability that the point will fall within any very small area $dxdy$ whose middle point has the coordinates $x$ and $y$, will be

$$z = \frac{h_1 h_2 dx\,dy}{\pi}e^{-(h_1{}^2x^2+h_2{}^2y^2)}. \tag{2}$$

The positions of the axes of $X$ and $Y$, and the values of $h_1$ and $h_2$, are to be found as in my recent article, ANALYST, Jan. to May '81. In this and other articles (Sept. '81, p. 145), it was shown that the functions (1) and (2) are the limiting forms of the series of coefficients in the expansion, to an infinitely high power, of a polynomial of one and two variables respectively. On similar principles, the law of errors in space of three dimensions may be obtained by the expansion of a polynomial of three variables, whose coefficients are all positive and their sum equal to unity. It is the object of the present paper to show this, and to demonstrate the limiting form of the expansion.

Taking at first the true position of the observed point as an origin, and drawing through it any three rectangular axes, we will refer to these axes any other positions which the point may take through errors of observation. The coordinates $x$, $y$, $z$ of any such point of error will be regarded as multiples of the arbitrary intervals or units of measure $\Delta x$, $\Delta y$ and $\Delta z$ respectively, which may be taken as small as need be. If for example a single observation places the point at $x = 4\Delta x$, $y = 2\Delta y$, $z = 3\Delta z$, we will write $L_{4,2,3}$ for the probability that this particular error will occur, and in general, we write $L_{a.b,c}$ for the probability of occurrence of the error $x = a\Delta x$, $y = b\Delta y$, $z = c\Delta z$, where $a$, $b$, $c$ are any whole numbers, positive or negative. Now take the probabilities $L$ as coefficients, and their sub-indices as exponents of $\xi$, $\eta$, $\zeta$, in the polynomial

$$u = \Sigma_{c=-m}^{c=m} \Sigma_{b=-m}^{b=m} \Sigma_{a=-m}^{a=m} (L_{a,\,b,\,c}\, \xi^a \eta^b \zeta^c). \tag{3}$$

Under this notation, the terms of the polynomial are supposed to be arranged in the form of a rectangular block, so that exponents and sub-indices vary from term to term by differences equal to unity. Each term is of the form included within the brackets, and the signs of summation denote that the whole polynomial is the sum of all the terms, and is formed by assigning to one of the exponents and sub-indices, $a$ for instance, all the integer values from $-m$ to $m$, while $b$ and $c$ remain constant, then in this sum assigning integer values to another exponent, $b$ for instance, from $-m$ to $m$, while $c$ remains constant, and finally in this sum assigning integer values to $c$ from $-m$ to $m$. In other words, the block-formed polynomial is the sum of all the different terms which can be formed by giving to $a$, $b$ and $c$, independently of each other, all the integer values from $-m$ to $m$. We might have adopted a notation analogous to that which I employed for polynomials of two variables, in ANALYST, Jan. '81, p. 4, but it would be quite cumbrous, if the eight corner terms of the block were written in full.

The number $m$ is supposed to be taken so large that no point of error will occur whose coordinates exceed the limits $x = \pm\, m\Delta x$, $y = \pm\, m\Delta y$, $z = \pm\, m\Delta z$. Any entire polynomial of three variables can be put in this block form, by adding or intercalating terms with zero coefficients if req'd. The whole number of terms in the block is $(2m + 1)^3$. From the manner in which the coefficients in the successive powers of a polynomial are formed, it is evident that when $n$ observations are made, the probability that the sum of the errors in the $x$ direction will be $s\Delta x$, and that at the same time the sums of those in the $y$ and $z$ directions will be $t\Delta y$ and $v\Delta z$ respectively, is the coefficient of $\xi^s\eta^t\zeta^v$ in the expansion of the polynomial (3) to the $n$th power.

If $L$ is the probability that a particular error or event will occur, in one observation, then in $n$ observations the most probable number of times for it to occur is $nL$. (ANALYST, May '79, p. 66.) This is strictly true when $nL$ is a whole number, and approximately true in other cases. Hence, the most probable total effect, in the $x$, $y$ and $z$ directions, of the possible error $4\Delta x$, $2\Delta y$, $3\Delta z$ for example, will be

$$nL_{4,\,2,\,3}(4\Delta x), \qquad nL_{4,\,2,\,3}(2\Delta y), \qquad nL_{4,\,2,\,3}(3\Delta z),$$

and in like manner for the other errors. The most probable algebraic sum of the $n$ errors which occur will be, in the $x$ direction,

$$n\Delta x\left[\Sigma_{c\,=\,-m}^{c\,=\,m}\Sigma_{b\,=\,-m}^{b\,=\,m}\Sigma_{a\,=\,-m}^{a\,=\,m}(aL_{a.\,b,\,c})\right], \tag{4}$$

where the coefficient $a$ takes the same values as the sub-index $a$ does. This is the approximate value of the exponent of $\xi$ in that term of the expansion of (3) whose coefficient is a maximum. The most probable value of the arithmetical mean of the $n$ errors which occur, in the $x$ direction, is found by dividing (4) by $n$. The quotient thus obtained is evidently the statical moment, with respect to the $YZ$ plane, of the system of coefficients $L$ regarded as the masses of material points. It also represents the lever arm of the system with respect to that plane, since $\Sigma L = 1$. In like manner it will be found that the most probable arith. means of the errors in the $Y$ and $Z$ directions, respectively, are the lever arms of the system with respect to the $XZ$ and $XY$ planes. Thus the most probable mean position of the $n$ points of error is the centre of gravity of the system of masses $L$. From what was shown in my article of May '80, p. 81, respecting the position of the centre of gravity of the coefficients in an expanded polynomial, joined with the fact that the coefficients in the expansion are not altered by subtracting a constant quantity from each exponent in the first power, so as to bring the place of $\xi^0\eta^0\zeta^0$ (or $x^0y^0z^0$ in the notation there used) into the middle of the block instead of at one corner, it follows that the lever arms of the system of coefficients in the expansion of (3) to the $n$th power are $n$ times what they are in the first power, the coordinate planes remaining the same. Hence, according to (4) &c., the maximum coefficient in the expansion coincides approximately with the centre of gravity of the whole system of coefficients in the expansion. It will otherwise appear hereafter, that this is true at the limit, when $n$ becomes infinite.

Let us write the expansion of (3) to the $n$th power as follows, retaining the same block form.

$$U = \Sigma_{c\,=\,-mn}^{c\,=\,mn}\Sigma_{b\,=\,-mn}^{b\,=\,mn}\Sigma_{a\,=\,-mn}^{a\,=\,mn}(l_{a,\,b,\,c}\,\xi^a\eta^b\zeta^c), \tag{5}$$

where $a$, $b$, $c$ take only integer values as before. The whole number of terms in the expansion is $(2mn+1)^3$. We shall now employ the method of indeterminate coefficients, to find a relation between the $(2m+1)^3$ coefficients $L$ in the first power, and any similar block of an equal number of coefficients $l$ in the expansion. Since the $L$ and $l$ are independent of $\xi$, $\eta$ and $\zeta$, we may take the relation

$$u^n = U, \tag{6}$$

and differentiate it with respect to the variables, obtaining

$$nu^{n-1}\left(\frac{du}{d\xi}\right) = \frac{dU}{d\xi}, \qquad nu^{n-1}\left(\frac{du}{d\eta}\right) = \frac{dU}{d\eta}, \qquad nu^{n-1}\left(\frac{du}{d\zeta}\right) = \frac{dU}{d\zeta},$$

which being divided by (6), gives

$$nU\left(\frac{du}{d\xi}\right) = u\left(\frac{dU}{d\xi}\right), \quad nU\left(\frac{du}{d\eta}\right) = u\left(\frac{dU}{d\eta}\right), \quad nU\left(\frac{du}{d\zeta}\right) = u\left(\frac{dU}{d\zeta}\right). \tag{7}$$

In the first of these three equations (7) substitute the expressions for $u$ and $U$ as in (3) and (5), and also expressions for $\frac{du}{d\xi}$ and $\frac{dU}{d\xi}$ as obtained from the same by differentiation. This gives an equation, each member of which contains the product of two block-formed polynomials. Any such product will itself be a block-formed polynomial, and any term in it will have for its coefficient the sum of products of $L$ and $l$, such that this sum can be expressed in the block notation. Forming in this way the coefficient of $\xi^{i-1}\eta^j\zeta^k$ in the product in each member of the equ'n referred to, and placing them equal to each other by the principle of indeterminate coefficients, we get

$$n\left[\Sigma_{c=-m}^{c=m}\Sigma_{b=-m}^{b=m}\Sigma_{a=-m}^{a=m}\left(-aL_{-a,-b,-c}\, l_{(i+a),(j+b),(k+c)}\right)\right]$$
$$=\Sigma_{c=-m}^{c=m}\Sigma_{b=-m}^{b=m}\Sigma_{a=-m}^{a=m}\left[(i+a)L_{-a,-b\ -c}\, l_{(i+a).(j+b),(k+c)}\right]. \tag{8}$$

In the second member of this equation, let the part which does not have the coefficient $i$ be transferred to the first member, then change the signs of both members and divide by $n+1$. Take $V$ as an auxiliary letter and write

$$V = \Sigma_{c=-m}^{c=m}\Sigma_{b=-m}^{b=m}\Sigma_{a=-m}^{a=m}\left(L_{-a,-b,-c}\, l_{(i+a),(j+b),(k+c)}\right). \tag{9}$$

Then (8) becomes

$$\sum_{c=-m}^{c=m}\sum_{b=-m}^{b=m}\sum_{a=-m}^{a=m}\left(aL_{-a,-b,-c}\, l_{(i+a),(j+b),(k+c)}\right) = \frac{-i}{n+1}V. \tag{10}$$

In like manner, starting from the two last equations in (7), and equating to each other the coefficients of $\xi^i\eta^{j-1}\zeta^k$ in the one case and those of $\xi^i\eta^j\zeta^{k-1}$ in the other, we get after reduction

$$\sum_{c=-m}^{c=m}\sum_{b=-m}^{b=m}\sum_{a=-m}^{a=m}\left(bL_{-a,-b,-c}\, l_{(i+a),(j+b),(k+c)}\right) = \frac{-j}{n+1}V, \tag{11}$$

$$\sum_{c=-m}^{c=m}\sum_{b=-m}^{b=m}\sum_{a=-m}^{a=m}\left(cL_{-a,-b,-c}\, l_{(i+a),(j+b),(k+c)}\right) = \frac{-k}{n+1}V. \tag{12}$$

It will be seen that the terms in (10), (11) and (12) all contain products of $L$ and $l$, so taken that any $L$ occupying a given position in its block, is the multiplier of that $l$ which occupies just the opposite position in the equal block of terms in the expansion. These three equations express the relation which exists between the coefficients $L$ in the given polynomial, and any similar block of coefficients $l$ in the expansion to the $n$th power. The middle coefficient in this block is $l_{i,j,k}$, and its coordinates are $i\varDelta x$, $j\varDelta y$ and $k\varDelta z$, the expansion being referred to the same axes as the first power. The coefficients $l$ are also regarded as the masses of material points. When the exponent $n$ is made very large, or infinite, the masses $l$ attain their limiting values, and we denote by $w$ the limiting value of $l_{i,j,k}$. Supposing the p'ts to be set so close together as to become consecutive, $\varDelta x$, $\varDelta y$, $\varDelta z$ are represented by $dx$, $dy$ and $dz$, the coordinates of the point whose mass is $w$ are

$$x = idx, \qquad y = jdy, \qquad z = kdz, \tag{13}$$

and we wish to express $w$ as a function of $x$, $y$ and $z$. Let $n$ be made an infinity of the second order. The system of material p'ts will then extend throughout infinite space, their number $(2mn+1)^3$ being the whole number of consecutive points in the infinite block, while $(2m+1)^3$ is the number included within the small block under consideration which has $w$ at its centre, so that the portion of space occupied by this last block is infinitesimal in its length, breadth and depth. The masses of the consecutive points within its limits may therefore be regarded as varying from each other by constant differences, under the assumption, of course, that $w$ is a continuous function of $x$, $y$, $z$.

Let $d_xw$, $d_yw$ and $d_zw$ denote the differentials of $w$ taken in the three coordinate directions. The values of all the $(2m+1)^3$ masses $l$ in the block will be formed from $w$ by successive additions and subtractions of its differentials, and will be the terms in this block-formed total,

$$\Sigma_{c=-m}^{c=m}\Sigma_{b=-m}^{b=m}\Sigma_{a=-m}^{a=m}(w+ad_xw+bd_yw+cd_zw). \tag{14}$$

Substitute them for the correspond'g terms $l$ in (10), (11) and (12). Collect separately the coefficients of $w$, $d_xw$, $d_yw$ and $d_zw$ in the result, remembering that $\Sigma L = 1$. Employ auxiliary letters, such that $\alpha_1$, $\alpha_2$, $\alpha_3$ denote the three sums of all the products formed by multiplying each $L$ by its first, second, and third sub-index respectively; also let $\beta_1$, $\beta_2$, $\beta_3$ denote the three sums of the products of each $L$ into the square of its first, second, and third sub-index respectively; and let $\gamma_1$, $\gamma_2$, $\gamma_3$ denote the three sums of the products formed by multiplying each $L$ by the product of its first and second, first and third, and second and third sub-indices respectively. It will be found that (10), (11) and (12) are then reducible to

$$\left.\begin{aligned}-\alpha_1 w+\beta_1 d_x w+\gamma_1 d_y w+\gamma_2 d_z w &= \frac{-i}{n+1}(w-\alpha_1 d_x w-\alpha_2 d_y w-\alpha_3 d_z w),\\ -\alpha_2 w+\gamma_1 d_x w+\beta_2 d_y w+\gamma_3 d_z w &= \frac{-j}{n+1}(w-\alpha_1 d_x w-\alpha_2 d_y w-\alpha_3 d_z w),\\ -\alpha_3 w+\gamma_2 d_x w+\gamma_3 d_y w+\beta_3 d_z w &= \frac{-k}{n+1}(w-\alpha_1 d_x w-\alpha_2 d_y w-\alpha_3 d_z w).\end{aligned}\right\} \quad (15)$$

These are differential equations in which $\alpha_1$, $\beta_1$ &c. are constants. From the manner in which the latter are formed, it is evident that if the coefficients $L$ of the given polynomial are regarded as the masses of material points, the intervals between them being $\Delta x$, $\Delta y$, $\Delta z$, represented by $dx$, $dy$, $dz$ at the limit, then $\alpha_1 dx$, $\alpha_2 dy$, $\alpha_3 dz$ are the statical moments of the system with respect to the coordinate planes $YZ$, $XZ$ and $XY$. Also $\beta_1(dx)^2$, $\beta_2(dy)^2$ and $\beta_3(dz)^2$ are the three sums of the products formed by multiplying each $L$ into the square of its distance from the three planes respectively. Likewise $\gamma_1 dxdy$, $\gamma_2 dxdz$ and $\gamma_3 dydz$ are the three sums of the products formed by multiplying each $L$ into the product of its distances from the planes $YZ$ and $XZ$, $YZ$ and $XY$, and $XZ$ and $XY$ respectively. It is evident that if the coefficients $L$ in the given polynomial (3) are such that their centre of gravity falls at the origin, that is, at the place of $L_{0,0,0}$, the statical moment of the system with respect to any plane through this centre will be null, and we shall have $\alpha_1=0$, $\alpha_2=0$, $\alpha_3=0$. Furthermore, if the coefficients and assumed coordinate axes are such that these axes are the free axes of the system of masses $L$, that is, the principal axes through the centre of gravity, we shall have also $\gamma_1=0$, $\gamma_2=0$, $\gamma_3=0$. This follows from the known properties of the principal axes in bodies of three dimensions. (See for instance Chapter II. of Vol. II. of Poisson's *Traite' de Me'canique.*)

Of course, if the possible points of error and their probabilities are arbitrarily distributed, in other words, if any arbitrary set of values is assigned to the coefficients $L$ in (3), their centre of gravity will not in general coincide with the origin or place of $L_{0,0,0}$, and the axes taken will not in general be free axes. But the position of the free axes can always be found by means of the formulas demonstrated in Poisson's treatise, and being at right angles to each other, they can be taken as coordinate axes. By reasoning precisely analogous to that employed in my article on errors in two dimensions (ANALYST, March '81, pp. 44 to 47), it will appear that when the coefficients $L$ are referred to any new rectangular axes, and the coefficients $l$ in the expansion are also referred to the same axes, the coefficients in the expansion are unchanged and retain the same positions relatively to each other which they had under the old system, differing only in position relatively to the axes.

For the sake of simplicity we will assume the same arbitrary interval or unit of measure in all directions, so that $\varDelta x = \varDelta y = \varDelta z$. A given system of coefficients located in space will be here said to be *referred to* any given rectangular axes, when the variables $\xi$, $\eta$, $\zeta$ attached to each coefficient receive exponents equal to the three coordinates of the coefficient, expressed in units of measure. And coefficients whose attached variables have any given exponents are said to be referred to given axes, when they are so placed in space that their coordinates, expressed in the units of measure, are equal to the corresponding exponents.

Suppose that the directions of the axes in (3) are changed while the origin remains the same. Let $x_1$, $y_1$, $z_1$ and $x_2$, $y_2$, $z_2$ be the coordinates, in units of measure, of two given points of error referred to the old axes, and let $x'_1$, $y'_1$, $z'_1$ and $x'_2$, $y'_2$, $z'_2$ be those of the same points referred to the new axes. Multiplying together the two corresponding terms of the polynomial, omitting their coefficients $L$ since the products of these have the same value in both cases, we find in the two systems

$$\left.\begin{aligned} \xi^{x_1}\eta^{y_1}\zeta^{z_1} \times \xi^{x_2}\eta^{y_2}\zeta^{z_2} &= \xi^{x_1+x_2}\eta^{y_1+y_2}\zeta^{z_1+z_2}, \\ \xi^{x'_1}\eta^{y'_1}\zeta^{z'_1}\times\xi^{x'_2}\eta^{y'_2}\zeta^{z'_2} &= \xi^{x'_1+x'_2}\eta^{y'_1+y'_2}\zeta^{z'_1+z'_2}. \end{aligned}\right\} \qquad (16)$$

The exponents of the second members are the coordinates of the place of the coefficient of the product of the two terms, in the two systems. The known relations between the two systems of coordinates give

$$\left.\begin{aligned} x_1 &= a\ x'_1+b\ y'_1+c\ z'_1, & x_2 &= a\ x'_2+b\ y'_2+c\ z'_2, \\ y_1 &= a'x'_1+b'y'_1+c'z'_1, & y_2 &= a'x'_2+b'y'_2+c'z'_2, \\ z_1 &= a''x'_1+b''y'_1+c''z'_1, & z_2 &= a''x'_2+b''y'_2+c''z'_2, \end{aligned}\right\} \quad (17)$$

where $a$, $b$, $c$ &c. are the direction-cosines of the new axes, that is to say, $a$, $b$, $c$ are the cosines of the angles which the new $X$, $Y$ and $Z$ axes make with the old $X$ axis; $a'$, $b'$, $c'$ are those of the angles which they make with the $Y$ axis; and so on. By addition we get

$$\left.\begin{aligned} x_1+x_2 &= a\ (x'_1+x'_2)+b\ (y'_1+y'_2)+c\ (z'_1+z'_2), \\ y_1+y_2 &= a'(x'_1+x'_2)+b'(y'_1+y'_2)+c'(z'_1+z'_2), \\ z_1+z_2 &= a''(x'_1+x'_2)+b''(y'_1+y'_2)+c''(z'_1+z'_2). \end{aligned}\right\} \qquad (18)$$

These relations, being of the same kind as those in (17), show that just as $x_1$, $y_1$, $z_1$ and $x'_1$, $y'_1$, $z'_1$ are coordinates of the same point in the two systems, so the point whose coordinates in the old system are $x_1+x_2$, $y_1+y_2$, $z_1+z_2$, is the same point whose coordinates in the new system are $x'_1+x'_2$, $y'_1+y'_2$, $z'_1+z'_2$. In other words, the product keeps the same absolute position under either system. And as this is true for all the partial products which make up the total, it follows that whatever new directions we give to the rectangular axes, the origin being unchanged, the absolute places of the coefficients in the squared polynomial will be the same.

In like manner it may be shown that they will continue the same for higher powers, and up to the limit. If we also transfer the origin to some new point, and draw axes through it parallel to the new directions as above, and denote by $a_1$, $b_1$, $c_1$ the coordinates of the old origin referred to the new, then in the first power of the polynomial all the exponents of $\xi$, $\eta$, $\zeta$ will be increased by the constant quantities $a_1$, $b_1$, $c_1$ respectively, in the second power they will all be increased by $2a_1$, $2b_1$, $2c_1$ respectively, and in the $n$th power, by $na_1$, $nb_1$, $nc_1$ respectively. Thus the positions of the coefficients in the expansion, relatively to each other, will be unaffected by a change in the origin, or in the directions of the axes, or in both together.

Supposing now that all the exponents and corresponding sub-indices in the polynomial (3) have been altered so as to refer to the new axes, it is evident that they will in general be fractional numbers, and $i$, $j$, and $k$ likewise, whereas we regarded them as integers, in our demonstration of the results (10), (11), (12). But we may imagine that the given block of coefficients $L$ in (3) is enclosed within a cubical block whose centre is at the desired new origin and whose sides are parallel to the new axes, and that this new block is subdivided into very small cubes by equidistant planes parallel to its sides and taken very near together.

The points at which these planes intersect are supposed to be each occupied by a coefficient $L$, only all these coefficients are zero except at the points which come nearest to the coefficients in the original block; and to these points the original coefficients are supposed to be transferred. It is evident that by diminishing the intervals between the planes, these positions of the coefficients $L$ in the new block can be made to approach to their true positions in the old block, within any assignable limits. Hence the relations (10), (11), (12) will hold good, it being understood that $m$ is now the number of new intervals from the centre to the surface of the new block, in any one of the three coordinate directions, and that $i, j, k$ denote the number of such intervals from the centre of any equal block of coefficients in the expansion, to the respective coordinate planes. But we may restore, if we please, the larger unit of measure $\Delta x = \Delta y = \Delta z$, by dividing (10), (11) and (12) throughout by the ratio which the small interval or unit bears to the larger one. The effect of this will be, that the coefficients from $m$ to $-m$, as well as $i$, $j$, $k$, will now, in general, be fractional numbers. The same is true of the coefficients from $m$ to $-m$ in (14). Here $d_xw$, $d_yw$ and $d_zw$ are the increments of $w$ corresponding to those increments of $x$, $y$ and $z$ which are equal to $\Delta x = \Delta y = \Delta z$, and which are represented by $dx = dy = dz$ at the limit.

[To be continued.]